On the Impossibility of Informationally Efficient Markets

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If competitive equilibrium is defined as a situation in which prices are such that all arbitrage profits are eliminated, is it possible that a competitive economy always be in equilibrium? Clearly not, for then those who arbitrage make no (private) return from their (privately) costly activity. Hence the assumptions that all markets, including that for information, are always in equilibrium and always perfectly arbitraged are inconsistent when arbitrage is costly.

We propose here a model in which there is an equilibrium degree of disequilibrium: prices reflect the information of informed individuals (arbitrageurs) but only partially, so that those who expend resources to obtain information do receive compensation. How informative the price system is depends on the number of individuals who are informed; but the number of individuals who are informed is itself an endogenous variable in the model.

The model is the simplest one in which prices perform a well-articulated role in conveying information from the informed to the uninformed. When informed individuals observe information that the return to a security is going to be high, they bid its price up, and conversely when they observe information that the return is going to be low. Thus the price system makes publicly available the information obtained by informed individuals to the uniformed. In general, however, it does this imperfectly; this is perhaps lucky, for were it to do it perfectly, an equilibrium would not exist.

In the introduction, we shall discuss the general methodology and present some conjectures concerning certain properties of the equilibrium. The remaining analytic sections of the paper are devoted to analyzing in detail an important example of our general model, in which our conjectures concerning the nature of the equilibrium can be shown to be correct. We conclude with a discussion of the implications of our approach and results, with particular emphasis on the relationship of our results to the literature on "efficient capital markets."

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I. The Model

Our model can be viewed as an extension of the noisy rational expectations model introduced by Robert Lucas and applied to the study of information flows between traders by Jerry Green (1973); Grossman (1975, 1976, 1978); and Richard Kihlstrom and Leonard Mirman. There are two assets: a safe asset yielding a return R, and a risky asset, the return to which, u, varies randomly from period to period. The variable u consists of two parts,

$$u = \theta + \varepsilon \tag{1}$$

where θ is observable at a cost c, and ε is unobservable.[1] Both θ and ε are random variables. There are two types of individuals, those who observe θ (informed traders), and those who observe only price (uninformed traders). In our simple model, all individuals are, *ex ante*, identical; whether they are informed or uninformed just depends on whether they have spent c to obtain information. Informed traders' demands will depend on θ and the price of the risky asset P. Uninformed traders' demands

[1]An alternative interpretation is that θ is a "measurement" of u with error. The mathematics of this alternative interpretation differ slightly, but the results are identical.

will depend only on P, but we shall assume that they have rational expectations; they learn the relationship between the distribution of return and the price, and use this in deriving their demand for the risky assets. If x denotes the supply of the risky asset, an equilibrium when a given percentage, λ, of traders are informed, is thus a price function $P_\lambda(\theta,x)$ such that, when demands are formulated in the way described, demand equals supply. We assume that uninformed traders do not observe x. Uninformed traders are prevented from learning θ via observations of $P_\lambda(\theta,x)$ because they cannot distinguish variations in price due to changes in the informed trader's information from variations in price due to changes in aggregate supply. Clearly, $P_\lambda(\theta,x)$ reveals some of the informed trader's information to the uninformed traders.

We can calculate the expected utility of the informed and the expected utility of the uninformed. If the former is greater than the latter (taking account of the cost of information), some individuals switch from being uninformed to being informed (and conversely). An overall equilibrium requires the two to have the same expected utility. As more individuals become informed, the expected utility of the informed falls relative to the uninformed for two reasons:

(a) The price system becomes more informative because variations in θ have a greater effect on aggregate demand and thus on price when more traders observe θ. Thus, more of the information of the informed is available to the uninformed. Moreover, the informed gain more from trade with the uninformed than do the uninformed. The informed, on average, buy securities when they are "underpriced" and sell them when they are "overpriced" (relative to what they would have been if information were equalized).[2] As the price system becomes more informative, the difference in their information—and hence the magnitude by which the informed can gain relative to the uninformed—is reduced.

(b) Even if the above effect did not occur, the increase in the ratio of informed to uninformed means that the relative gains of the informed, on a per capita basis, in trading with the uninformed will be smaller.

We summarize the above characterization of the equilibrium of the economy in the following two conjectures:

Conjecture 1: The more individuals who are informed, the more informative is the price system.

Conjecture 2: The more individuals who are informed, the lower the ratio of expected utility of the informed to the uninformed.

(Conjecture 1 obviously requires a definition of "more informative"; this is given in the next section and in fn. 7.)

The equilibrium number of informed and uninformed individuals in the economy will depend on a number of critical parameters: the cost of information, how informative the price system is (how much noise there is to interfere with the information conveyed by the price system), and how informative the information obtained by an informed individual is.

Conjecture 3: The higher the cost of information, the smaller will be the equilibrium percentage of individuals who are informed.

Conjecture 4: If the quality of the informed trader's information increases, the more their demands will vary with their information and thus the more prices will vary with θ. Hence, the price system becomes more informative. The equilibrium proportion of informed to uninformed may be either increased or decreased, because even though the value of being informed has increased due to the increased quality of θ, the value of being uninformed has also increased because the price system becomes more informative.

Conjecture 5: The greater the magnitude of noise, the less informative will the price system be, and hence the lower the expected utility of uninformed individuals. Hence, in equilibrium the greater the magnitude of noise, the larger the proportion of informed individuals.

[2]The framework described herein does not explicitly model the effect of variations in supply, i.e., x on commodity storage. The effect of futures markets and storage capabilities on the informativeness of the price system was studied by Grossman (1975, 1977).

Conjecture 6: In the limit, when there is no noise, prices convey all information, and there is no incentive to purchase information. Hence, the only possible equilibrium is one with no information. But if everyone is uninformed, it clearly pays some individual to become informed.[3] Thus, there does not exist a competitive equilibrium.[4]

Trade among individuals occurs either because tastes (risk aversions) differ, endowments differ, or beliefs differ. This paper focuses on the last of these three. An interesting feature of the equilibrium is that beliefs may be precisely identical in either one of two situations: when all individuals are informed or when all individuals are uninformed. This gives rise to:

Conjecture 7: That, other things being equal, markets will be thinner under those conditions in which the percentage of individuals who are informed (λ) is either near zero or near unity. For example, markets will be thin when there is very little noise in the system (so λ is near zero), or when costs of information are very low (so λ is near unity).

In the last few paragraphs, we have provided a number of conjectures describing the nature of the equilibrium when prices convey information. Unfortunately, we have not been able to obtain a general proof of any of these propositions. What we have been able to do is to analyze in detail an interesting example, entailing constant absolute risk-aversion utility functions and normally distributed random variables. In this example, the equilibrium price distribution can actually be calculated, and all of the conjectures provided above can be verified. The next sections are devoted to solving for the equilibrium in this particular example.[5]

II. Constant Absolute Risk-Aversion Model

A. *The Securities*

The ith trader is assumed to be endowed with stocks of two types of securities: $\overline{M}_i$, the riskless asset, and $\overline{X}_i$, a risky asset. Let P be the current price of risky assets and set the price of risk free assets equal to unity. The ith trader's budget constraint is

$$PX_i + M_i = W_{0i} \equiv \overline{M}_i + P\overline{X}_i \tag{2}$$

Each unit of the risk free asset pays R "dollars" at the end of the period, while each unit of the risky asset pays u dollars. If at the end of the period, the ith trader holds a portfolio (M_i, X_i), his wealth will be

$$W_{1i} = RM_i + uX_i \tag{3}$$

B. *Individual's Utility Maximization*

Each individual has a utility function $V_i(W_{1i})$. For simplicity, we assume all individuals have the same utility function and so drop the subscripts i. Moreover, we assume the utility function is exponential, i.e.,

$$V(W_{1i}) = -e^{-aW_{1i}}, \qquad a > 0$$

where a is the coefficient of absolute risk aversion. Each trader desires to maximize expected utility, using whatever information is available to him, and to decide on what information to acquire on the basis of the consequences to his expected utility.

Assume that in equation (1) θ and ε have a multivariate normal distribution, with

$$E\varepsilon = 0 \tag{4}$$

$$E\theta\varepsilon = 0 \tag{5}$$

$$Var(u^*|\theta) = Var\,\varepsilon^* \equiv \sigma_\varepsilon^2 > 0 \tag{6}$$

[3]That is, with no one informed, an individual can only get information by paying c dollars, since no information is revealed by the price system. By paying c dollars an individual will be able to predict better than the market when it is optimal to hold the risky asset as opposed to the risk-free asset. Thus his expected utility will be higher than an uninformed person gross of information costs. Thus for c sufficiently low all uninformed people will desire to be informed.

[4]See Grossman (1975, 1977) for a formal example of this phenomenon in futures markets. See Stiglitz (1971, 1974) for a general discussion of information and the possibility of nonexistence of equilibrium in capital markets.

[5]The informational equilibria discussed here may not, in general, exist. See Green (1977). Of course, for the utility function we choose equilibrium does exist.

since θ and ε are uncorrelated. Throughout this paper we will put a * above a symbol to emphasize that it is a random variable. Since W_{1i} is a linear function of ε, for a given portfolio allocation, and a linear function of a normally distributed random variable is normally distributed, it follows that W_{1i} is normal conditional on θ. Then, using (2) and (3) the expected utility of the *informed* trader with information θ can be written

$$(7) \quad E(V(W_{1i}^*)|\theta) = -\exp\left(-a\left\{E[W_{1i}^*|\theta] - \frac{a}{2} Var[W_{1i}^*|\theta]\right\}\right)$$

$$= -\exp\left(-a\left[RW_{0i} + X_I\{E(u^*|\theta) - RP\} - \frac{a}{2} X_I^2 Var(u^*|\theta)\right]\right)$$

$$= -\exp\left(-a\left[RW_{0i} + X_I(\theta - RP) - \frac{a}{2} X_I^2 \sigma_\varepsilon^2\right]\right)$$

where X_I is an informed individual's demand for the risky security. Maximizing (7) with respect to X_I yields a demand function for risky assets:

$$(8) \qquad X_I(P,\theta) = \frac{\theta - RP}{a\sigma_\varepsilon^2}$$

The right-hand side of (8) shows the familiar result that with constant absolute risk aversion, a trader's demand does not depend on wealth; hence the subscript i is not on the left-hand side of (8).

We now derive the demand function for the uninformed. Let us assume the only source of "noise" is the per capita supply of the risky security x.

Let $P^*(\cdot)$ be some particular price function of (θ, x) such that u^* and P^* are jointly normally distributed. (We will prove that this exists below.)

Then, we can write for the uninformed individual

$$(7') \quad E(V(W_{1i}^*)|P^*) = -\exp\left[-a\left\{E[W_{1i}^*|P^*] - \frac{a}{2} Var[W_{1i}^*|P^*]\right\}\right]$$

$$= -\exp\left[-a\left\{RW_{0i} + X_U(E[u^*|P^*] - RP) - \frac{a}{2} X_U^2 Var[u^*|P^*]\right\}\right]$$

The demands of the uninformed will thus be a function of the price function P^* and the actual price P.

$$(8') \quad X_U(P;P^*) = \frac{E[u^*|P^*(\theta,x) = P] - RP}{a\, Var[u^*|P^*(\theta,x) = P]}$$

C. *Equilibrium Price Distribution*

If λ is some particular fraction of traders who decide to become informed, then define an equilibrium price system as a function of $(\theta, x), P_\lambda(\theta, x)$, such that for all (θ, x) per capita demands for the risky assets equal supplies:

$$(9) \qquad \lambda X_I(P_\lambda(\theta,x),\theta) + (1-\lambda) X_U(P_\lambda(\theta,x); P_\lambda^*) = x$$

The function $P_\lambda(\theta, x)$ is a statistical equilibrium in the following sense. If over time uninformed traders observe many realizations of (u^*, P_λ^*), then they learn the joint distribution of (u^*, P_λ^*). After all learning about the joint distribution of (u^*, P_λ^*) ceases, all traders will make allocations and form expectations such that this joint distribution persists over time. This follows from (8), (8′), and (9), where the market-clearing price that comes about is the one which takes into account the fact that uninformed traders have learned that it contains information.

We shall now prove that there exists an equilibrium price distribution such that P^* and u^* are jointly normal. Moreover, we shall be able to characterize the price distribution. We define

$$(10a)\quad w_\lambda(\theta,x)=\theta-\frac{a\sigma_\varepsilon^2}{\lambda}(x-Ex^*)$$

for $\lambda>0$, and define $w_0(\theta,x)$ as the number:

$$(10b)\quad w_0(\theta,x)=x \qquad \text{for all } (\theta,x)$$

where w_λ is just the random variable θ, plus noise.[6] The magnitude of the noise is inversely proportional to the proportion of informed traders, but is proportional to the variance of ε. We shall prove that the equilibrium price is just a linear function of w_λ. Thus, if $\lambda>0$, the price system conveys information about θ, but it does so imperfectly.

D. *Existence of Equilibrium and a Characterization Theorem*

THEOREM 1: *If $(\theta^*,\varepsilon^*,x^*)$ has a nondegenerate joint normal distribution such that θ^*, ε^*, and x^* are mutually independent, then there exists a solution to* (9) *which has the form $P_\lambda(\theta,x)=\alpha_1+\alpha_2 w_\lambda(\theta,x)$, where α_1 and α_2 are real numbers which may depend on λ, such that $\alpha_2>0$. (If $\lambda=0$, the price contains no information about θ.) The exact form of $P_\lambda(\theta,x)$ is given in equation* (A10) *in Appendix B. The proof of this theorem is also in Appendix B.*

The importance of Theorem 1 rests in the simple characterization of the information in the equilibrium price system: P_λ^* is informationally equivalent to w_λ^*. From (10) w_λ^* is a "mean-preserving spread" of θ; i.e., $E[w_\lambda^*|\theta]=\theta$ and

$$(11)\qquad Var[w_\lambda^*|\theta]=\frac{a^2\sigma_\varepsilon^4}{\lambda^2}\,Var\,x^*$$

For each replication of the economy, θ is the information that uninformed traders would like to know. But the noise x^* prevents w_λ^* from revealing θ. How well-informed uninformed traders can become from observing P_λ^* (equivalently w_λ^*) is measured by $Var[w_\lambda^*|\theta]$. When $Var[w_\lambda^*|\theta]$ is zero, w_λ^* and θ are perfectly correlated. Hence when uninformed firms observe w_λ^*, this is equivalent to observing θ. On the other hand, when $Var[w_\lambda^*|\theta]$ is very large, there are "many" realizations of w_λ^* that are associated with a given θ. In this case the observation of a particular w_λ^* tells very little about the actual θ which generated it.[7]

From equation (11) it is clear that large noise (high $Var\,x^*$) leads to an imprecise price system. The other factor which determines the precision of the price system $(a^2\sigma_\varepsilon^4/\lambda^2)$ is more subtle. When a is small (the individual is not very risk averse) or σ_ε^2 is small (the information is very precise), an informed trader will have a demand for risky assets which is very responsive to changes in θ. Further, the larger λ is, the more responsive is the total demand of informed traders. Thus small $(a^2\sigma_\varepsilon^4/\lambda^2)$ means that the aggregate demand of informed traders is very responsive to θ. For a fixed amount of noise (i.e., fixed $Var\,x^*$) the larger are the movements in aggregate demand which are due to movements in θ, the more will price movements be due to movements in θ. That is, x^* becomes less important relative to θ in determining price movements. Therefore, for small $(a^2\sigma_\varepsilon^4/\lambda^2)$ uninformed traders are able to confidently know that price is, for example, unusually high due to θ being high. In this way information from informed traders is transferred to uninformed traders.

[6]If $y'=y+Z$, and $E[Z|y]=0$, then y' is just y plus noise.

[7]Formally, w_λ^* is an experiment in the sense of Blackwell which gives information about θ. It is easy to show that, *ceteris paribus*, the smaller $Var(w_\lambda^*|\theta)$ the more "informative" (or sufficient) in the sense of Blackwell, is the experiment; see Grossman, Kihlstrom, and Mirman (p. 539).

E. *Equilibrium in the Information Market*

What we have characterized so far is the equilibrium price distribution for given λ. We now define an *overall* equilibrium to be a pair (λ, P_λ^*) such that the expected utility of the informed is equal to that of the uninformed if $0<\lambda<1$; $\lambda=0$ if the expected utility of the informed is less than that of the uninformed at P_0^*; $\lambda=1$ if the expected utility of the informed is greater than the uninformed at P_1^*. Let

$$\text{(12a)}\quad W_{Ii}^\lambda \equiv R(W_{0i}-c) + [u - RP_\lambda(\theta,x)]X_I(P_\lambda(\theta,x),\theta)$$

$$\text{(12b)}\quad W_{U1}^\lambda \equiv RW_{0i} + [\dot{u} - RP_\lambda(\theta,x)]X_U(P_\lambda(\theta,x);P_\lambda^*)$$

where c is the cost of observing a realization of θ^*. Equation (12a) gives the end of period wealth of a trader if he decides to become informed, while (12b) gives his wealth if he decides to be uninformed. Note that end of period wealth is random due to the randomness of W_{0i}, u, θ, and x.

In evaluating the expected utility of W_{Ii}^λ, we do not assume that a trader knows which realization of θ^* he gets to observe if he pays c dollars. A trader pays c dollars and then gets to observe some realization of θ^*. The overall expected utility of W_{Ii}^λ averages over all possible θ^*, ε^*, x^*, and W_{0i}. The variable W_{0i} is random for two reasons. First from (2) it depends on $P_\lambda(\theta,x)$, which is random as (θ,x) is random. Secondly, in what follows we will assume that $\overline{X}_i$ is random.

We will show below that $EV(W_{Ii}^\lambda)/EV(W_{Ui}^\lambda)$ is independent of i, but is a function of λ, a, c, and σ_ε^2. More precisely, in Appendix B we prove

THEOREM 2: *Under the assumptions of Theorem* 1, *and if $\overline{X}_i$ is independent of (u^*,θ^*,x^*) then*

$$\text{(13)}\quad \frac{EV(W_{Ii}^\lambda)}{EV(W_{Ui}^\lambda)} = e^{ac}\sqrt{\frac{Var(u^*|\theta)}{Var(u^*|w_\lambda)}}$$

F. *Existence of Overall Equilibrium*

Theorem 2 is useful, both in proving the uniqueness of overall equilibrium and in analyzing comparative statics. Overall equilibrium, it will be recalled, requires that for $0<\lambda<1$, $EV(W_{Ii}^\lambda)/EV(W_{Ui}^\lambda)=1$. But from (13)

$$\text{(14)}\quad \frac{EV(W_{Ii}^\lambda)}{EV(W_{Ui}^\lambda)} = e^{ac}\sqrt{\frac{Var(u^*|\theta)}{Var(u^*|w_\lambda)}} \equiv \gamma(\lambda)$$

Hence overall equilibrium simply requires, for $0<\lambda<1$,

$$\text{(15)}\quad \gamma(\lambda)=1$$

More precisely, we now prove

THEOREM 3: *If* $0\leqslant\lambda\leqslant 1$, $\gamma(\lambda)=1$, *and* P_λ^* *is given by* (A10) *in Appendix B, then* (λ,P_λ^*) *is an overall equilibrium. If* $\gamma(1)<1$, *then* $(1,P_1^*)$ *is an overall equilibrium. If* $\gamma(0)>1$, *then* $(0,P_0^*)$ *is an overall equilibrium. For all price equilibria* P_λ *which are monotone functions of* w_λ, *there exists a unique overall equilibrium* (λ,P_λ^*).

PROOF:

The first three sentences follow immediately from the definition of overall equilibrium given above equation (12), and Theorems 1 and 2. Uniqueness follows from the monotonicity of $\gamma(\cdot)$ which follows from (A11) and (14). The last two sentences in the statement of the theorem follow immediately.

In the process of proving Theorem 3, we have noted

COROLLARY 1: $\gamma(\lambda)$ is a strictly monotone increasing function of λ.

This looks paradoxical; we expect the ratio of informed to uninformed expected utility to be a decreasing function of λ. But, *we have defined utility as negative*. Therefore as λ rises, the expected utility of informed traders does go down relative to uninformed traders.

Note that the function $\gamma(0)=e^{ac}(Var(u^*|\theta)/Var\, u^*)^{1/2}$. Figure 1 illustrates the determination of the equilibrium λ. The figure assumes that $\gamma(0)<1<\gamma(1)$.

G. *Characterization of Equilibrium*

We wish to provide some further characterization of the equilibrium. Let us define

$$m=\left(\frac{a\sigma_\varepsilon^2}{\lambda}\right)^2\frac{\sigma_x^2}{\sigma_\theta^2} \tag{16a}$$

$$n=\frac{\sigma_\theta^2}{\sigma_\varepsilon^2} \tag{16b}$$

Note that m is inversely related to the informativeness of the price system since the squared correlation coefficient between P_λ^* and θ^*, ρ_θ^2 is given by

$$\rho_\theta^2=\frac{1}{1+m} \tag{17}$$

Similarly, n is directly related to the quality of the informed trader's information because $n/(1+n)$ is the squared correlation coefficient between θ^* and u^*.

Equations (14) and (15) show that the cost of information c, determines the equilibrium ratio of information quality between informed and uninformed traders $(Var(u^*|\theta))/Var(u^*|w_\lambda)$. From (1), (A11) of Appendix A, and (16), this can be written as

(18)

$$\frac{Var(u^*|\theta)}{Var(u^*|w_\lambda)}=\frac{1+m}{1+m+nm}=\left(1+\frac{nm}{1+m}\right)^{-1}$$

Substituting (18) into (14) and using (15) we obtain, for $0<\lambda<1$, in equilibrium

$$m=\frac{e^{2ac}-1}{1+n-e^{2ac}} \tag{19a}$$

or

$$1-\rho_\theta^2=\frac{e^{2ac}-1}{n} \tag{19b}$$

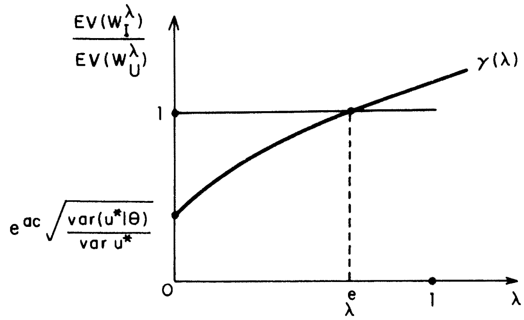

FIGURE 1

Note that (19) holds for $\gamma(0)<1<\gamma(1)$, since these conditions insure that the equilibrium λ is between zero and one. Equation (19b) shows that the equilibrium informativeness of the price system is determined completely by the cost of information c, the quality of the informed trader's information n, and the degree of risk aversion a.

H. *Comparative Statics*

From equation (19b), we immediately obtain some basic comparative statics results:

1) An increase in the quality of information (n) increases the informativeness of the price system.

2) A decrease in the cost of information increases the informativeness of the price system.

3) A decrease in risk aversion leads informed individuals to take larger positions, and this increases the informativeness of the price system.

Further, all other changes in parameters, such that n, a, and c remain constant, do not change the equilibrium degree of informativeness of the price system; other changes lead only to particular changes in λ of a magnitude to exactly offset them. For example:

4) An increase in noise (σ_x^2) increases the proportion of informed traders. At any given λ, an increase in noise reduces the informativeness of the price system; but it increases the returns to information and leads more individuals to become informed; the remarkable result obtained above establishes that *the two effects exactly offset each*

other so that the equilibrium informativeness of the price system is unchanged. This can be illustrated diagrammatically if we note from (16a) that for a given λ, an increase in σ_x^2 raises m which from (18) lowers $(Var(u^*|\theta))/Var(u^*|w_\lambda)$. Thus from (14) a rise in σ_x^2 leads to a vertical downward shift of the $\gamma(\lambda)$ curve in Figure 1, and thus a higher value of λ^e.

5) Similarly an increase in σ_ε^2 for a constant n (equivalent to an increase in the variance of u since n is constant) leads to an increased proportion of individuals becoming informed—and indeed again just enough to offset the increased variance, so that the degree of *informativeness* of the price system remains unchanged. This can also be seen from Figure 1 if (16) is used to note that an increase in σ_ε^2 with n held constant by raising σ_θ^2 leads to an increase in m for a given λ. From (18) and (14) this leads to a vertical downward shift of the $\gamma(\lambda)$ curve and thus a higher value of λ^e.

6) It is more difficult to determine what happens if, say σ_θ^2 increases, keeping σ_u^2 constant (implying a fall in σ_ε^2), that is, *the information obtained is more informative*. This leads to an increase in n, which from (19b) implies that the equilibrium informativeness of the price system rises. From (16) it is clear that m and nm both fall when σ_θ^2 rises (keeping $\sigma_u^2=\sigma_\theta^2+\sigma_\varepsilon^2$ constant). This implies that the $\gamma(\lambda)$ curve may shift up or down depending on the precise values of c, a, and n.[8] This ambiguity arises because an improvement in the precision of informed traders' information, with the cost of the information fixed, increases the benefit of being informed. However, some of the improved information is transmitted, via a more informative price system, to the uninformed; this increases the benefits of being uninformed. If n is small, both the price system m is not very informative *and* the marginal value of information to informed traders is high. Thus the *relative* benefits of being informed rises when n rises; implying that the equilibrium λ rises. Conversely when n is large the price system is very informative and the marginal value of information is low to informed traders so the relative benefits of being uninformed rises.

7) From (14) it is clear that an increase in the cost of information c shifts the $\gamma(\lambda)$ curve up and thus decreases the percentage of informed traders.

The above results are summarized in the following theorem.

THEOREM 4: *For equilibrium λ such that $0<\lambda<1$:*

A. *The equilibrium informativeness of the price system, ρ_θ^2, rises if n rises, c falls, or a falls.*

B. *The equilibrium informativeness of the price system is unchanged if σ_x^2 changes, or if σ_u^2 changes with n fixed.*

C. *The equilibrium percentage of informed traders will rise if σ_x^2 rises, σ_u^2 rises for a fixed n, or c falls.*

D. *If $\bar{n}$ satisfies $(e^{2ac}-1)/(\bar{n}-(e^{2ac}-1))=\bar{n}/(\bar{n}+1)$, then $n \overset{(<)}{>} \bar{n}$ implies that λ falls (rises) due to an increase in n.*

PROOF:

Parts A−C are proved in the above remarks. Part D is proved in footnote 8.

I. *Price Cannot Fully Reflect Costly Information*

We now consider certain limiting cases, for $\gamma(0)\leq 1\leq\gamma(1)$, and show that equilibrium does not exist if $c>0$ and price is fully informative.

1) As the cost of information goes to zero, the price system becomes more infor-

[8]From (14) and (18) it is clear that λ rises if and only if $Var(u^*|\theta)+Var(u^*|w_\lambda)$ falls due to the rise in σ_θ^2 for a given λ. This occurs if and only if $nm/(1+m)$ rises. Using (16) to differentiate $nm/(1+m)$ with respect to σ_ε^2 subject to the constraint that $d\sigma_u^2=0$ (i.e., $d\sigma_\theta^2=-d\sigma_\varepsilon^2$), we find that the sign of

$$\frac{d}{d\sigma_\theta^2}\left(\frac{nm}{1+m}\right)=sgn\left[m\left(\frac{n+1}{n}\right)-1\right]$$

$$=sgn\left[\left(\frac{\gamma}{n-\gamma}\right)\left(\frac{n+1}{n}\right)-1\right]$$

where $\gamma\equiv e^{2ac}-1$ and the last equality follows from equation (19a). Thus for n very large the derivative is negative so that λ falls due to an increase in the precision of the informed trader's information. Similarly if n is sufficiently small, the derivative is positive and thus λ rises.

mative, but at a positive value of c, say $\hat{c}$, all traders are informed. From (14) and (15) $\hat{c}$ satisfies

$$e^{a\hat{c}}\sqrt{\frac{Var(u^*|\theta)}{Var(u^*|w_1)}}=1$$

2) From (19a) as the precision of the informed trader's information n goes to infinity, i.e., $\sigma_\varepsilon^2\to 0$ and $\sigma_\theta^2\to\sigma_u^2$, σ_u^2 held fixed, the price system becomes perfectly informative. Moreover the percentage of informed traders goes to zero! This can be seen from (18) and (15). That is, as $\sigma_\varepsilon^2\to 0$, $nm/(1+m)$ must stay constant for equilibrium to be maintained. But from (19b) and (17), m falls as σ_ε^2 goes to zero. Therefore nm must fall, but nm must not go to zero or else $nm/(1+m)$ would not be constant. From (16) $nm=(a/\lambda)^2\sigma_\varepsilon^2\sigma_x^2$, and thus λ must go to zero to prevent nm from going to zero as $\sigma_\varepsilon^2\to 0$.

3) From (16a) and (19a) it is clear that as noise σ_x^2 goes to zero, the percentage of informed traders goes to zero. Further, since (19a) implies that m does not change as σ_x^2 changes, the informativeness of the price system is unchanged as $\sigma_x^2\to 0$.

Assume that c is small enough so that it is worthwhile for a trader to become informed when no other trader is informed. Then if $\sigma_x^2=0$ or $\sigma_\varepsilon^2=0$, there exists no competitive equilibrium. To see this, note that equilibrium requires either that the ratio of expected utility of the informed to the uninformed be equal to unity, or that if the ratio is larger than unity, no one be informed. We shall show that when no one is informed, it is less than unity so that $\lambda=0$ cannot be an equilibrium; but when $\lambda>0$, it is greater than unity. That is, if $\sigma_x^2=0$ or $\sigma_\varepsilon^2=0$, the ratio of expected utilities is not a continuous function of λ at $\lambda=0$.

This follows immediately from observing that at $\lambda=0$, $Var(u^*|w_0)=Var\,u^*$, and thus by (14)

$$\frac{EV(W_{Ii}^0)}{EV(W_{Ui}^0)}=e^{ac}\sqrt{\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2+\sigma_\theta^2}}=e^{ac}\sqrt{\frac{1}{1+n}}\tag{20}$$

while if $\lambda>0$, by (18)

$$\frac{EV(W_{Ii}^\lambda)}{EV(W_{Ui}^\lambda)}=e^{ac}\sqrt{\frac{1}{1+n\dfrac{m}{m+1}}}$$

But if $\sigma_x^2=0$ or $\sigma_\varepsilon^2=0$, then $m=0$, $nm=0$ for $\lambda>0$, and hence

$$\lim_{\lambda\to 0}\frac{EV(W_{Ii}^\lambda)}{EV(W_{Ui}^\lambda)}=e^{ac}\tag{21}$$

It immediately follows that

THEOREM 5: (a) *If there is no noise* ($\sigma_x^2=0$), *an overall equilibrium does not exist if (and only if)* $e^{ac}<\sqrt{1+n}$. (b) *If information is perfect* ($\sigma_\varepsilon^2=0, n=\infty$), *there never exists an equilibrium.*

PROOF:

(a) If $e^{ac}<\sqrt{1+n}$, then by (20) and (21), $\gamma(\lambda)$ is discontinuous at $\lambda=0$; $\lambda=0$ is not an equilibrium since by (20) $\gamma(0)<1$; $\lambda>0$ is not an equilibrium since by (21) $\gamma(\lambda)>1$.

(b) If $\sigma_\varepsilon^2=0$ and $\sigma_\theta^2=\sigma_u^2$ so that information is perfect, then for $\lambda>0$, $nm=0$ by (16) and hence $\gamma(\lambda)>1$ by (21). From (20) $\gamma(0)=0<1$.

If there is no noise and some traders become informed, then *all* their information is transmitted to the uninformed by the price system. Hence each informed trader acting as a price taker thinks the informativeness of the price system will be unchanged if he becomes uninformed, so $\lambda>0$ is not an equilibrium. On the other hand, if no traders are informed, then each uninformed trader learns nothing from the price system, and thus he has a desire to become informed (if $e^{ac}<(1+n)^{1/2}$). Similarly if the informed traders get perfect information, then their demands are very sensitive to their information, so that the market-clearing price becomes very sensitive to their information and thus reveals θ to the uninformed. Hence all traders desire to be uninformed. But if all traders are uninformed, each trader can eliminate the risk of his portfolio by the purchase of information, so each trader desires to be informed.

In the next section we show that the nonexistence of competitive equilibrium can be thought of as the breakdown of competitive markets due to lack of trade. That is, we will show that as σ_x^2 gets very small, trade goes to zero and markets serve no function. Thus competitive markets close for lack of trade "before" equilibrium ceases to exist at $\sigma_x^2=0$.

III. On the Thinness of Speculative Markets

In general, trade takes place because traders differ in endowments, preferences, or beliefs. Grossman (1975, 1977, 1978) has argued that differences in preferences are not a major factor in explaining the magnitude of trade in speculative markets. For this reason the model in Section II gave all traders the same risk preferences (note that none of the results in Section II are affected by letting traders have different coefficients of absolute risk aversion). In this section we assume that trade requires differences in endowments or beliefs and dispense with differences in risk preference as an explanatory variable.[9]

There is clearly some fixed cost in operating a competitive market. If traders have to bear this cost, then trade in the market must be beneficial. Suppose traders have the same endowments and beliefs. Competitive equilibrium will leave them with allocations which are identical with their initial endowments. Hence, if it is costly to enter such a competitive market, no trader would ever enter. We will show below that in an important class of situations, there is continuity in the amount of net trade. That is, when initial endowments are the same and peoples' beliefs differ *slightly*, then the competitive equilibrium allocation that an individual gets will be only *slightly* different from his initial endowment. Hence, there will only be a slight benefit to entering the competitive market. This could, for sufficiently high operating costs, be outweighed by the cost of entering the market.

The amount of trade occurring at any date is a random variable; a function of θ and x. It is easy to show that it is a normally distributed random variable. Since one of the primary determinants of the size of markets is differences in beliefs, one might have conjectured that markets will be thin, in some sense, if almost all traders are either informed or uninformed. This is not, however, obvious, since the amount of trade by any single trader may be a function of λ as well, and a few active traders can do the job of many small traders. In our model, there is a sense, however, in which our conjecture is correct.

We first calculate the magnitude of trades as a function of the exogenous parameters, θ and x. Let $h\equiv\sigma_\varepsilon^2$, $\bar{x}=Ex^*$, and $\bar{\theta}\equiv E\theta^*$. (The actual trades will depend on the distribution of random endowments across all of the traders, but these we shall net out.) Per capita net trade is[10]

$$(22)\quad X_I-x=(1-\lambda)\Big[\Big(nm+\frac{ah}{\lambda}\Big)(x-\bar{x})+\big[(m+1)n-1\big](\theta-\bar{\theta})+\bar{x}nm\Big]\div\big[1+m+\lambda nm\big]$$

[9]In the model described in Section II it was assumed that an individual's endowment $\bar{X}_i$ is independent of the market's per capita endowment x^*. This was done primarily so there would not be useful information in an individual's endowment about the total market endowment. Such information would be useful in equilibrium because an individual observes $P_\lambda(\theta,x)$. If due to observing $\bar{X}_i$, he knows something about x, then by observing $P_\lambda(\theta,x)$, $\bar{X}_i$ is valuable in making inferences about θ. To take this into account is possible, but would add undue complication to a model already overburdened with computations.

[10]Calculation of distribution of net trades

$$\frac{\lambda}{ah}(\theta-RP_\lambda)+\frac{(1\text{-}\lambda)\Big[(\bar{\theta}-RP_\lambda)(1+m)n+\theta-\bar{\theta}-\frac{ah}{\lambda}(x-\bar{x})\Big]}{ah(1+m+nm)n}=x$$

or $$\frac{(\theta-RP_\lambda)}{ah}\Big(\lambda+\frac{(1-\lambda)(1+m)}{1+m+nm}\Big)$$

$$=\Big(\frac{\theta-RP_\lambda}{ah}\Big)\Big(\frac{1+m+\lambda nm}{1+m+nm}\Big)$$

$$=x+\frac{(1-\lambda)\Big([(m+1)n-1](\theta-\bar{\theta})+\frac{ah}{\lambda}(x-\bar{x})\Big)}{ah(1+m+\lambda nm)n}$$

Thus, the mean of total informed trade is

$$(23)\qquad E\lambda(X_I - x) = \frac{(1-\lambda)\lambda m\bar{x}}{1+m+\lambda nm}$$

and its variance is

$$(24)\qquad \sigma_\theta^2(1-\lambda)^2\lambda^2\left[\left[(m+1)n-1\right]^2 + \left(nm+\frac{a\sigma_\varepsilon^2}{\lambda}\right)^2\frac{\sigma_x^2}{\sigma_\theta^2}\right] \div (1+m+\lambda nm)^2 n^2$$

In the last section we considered limiting values of the exogenous variables with the property that $\lambda \to 0$. The following theorem will show that the mean and variance of trade go to zero as $\lambda \to 0$. That is, the distribution of $\lambda(X_I - x)$ becomes degenerate at zero as $\lambda \to 0$. This is not trivial because as $\lambda \to 0$ due to $n \to \infty$ (very precise information), the informed trader's demand $X_I(P,\theta)$ goes to infinity at most prices because the risky asset becomes riskless with perfect information.

THEOREM 6: (a) *For sufficiently large or small c, the mean and variance of trade is zero.* (b) *As the precision of informed traders' information n goes to infinity, the mean and variance of trade go to zero.*

PROOF:

(a) From remark 1) in Section II, Part I, $\lambda = 1$ if $c \leqslant \hat{c}$, which from (23) and (24) implies trade is degenerate at zero. From (14), for c sufficiently large, say c^0, $\gamma(0)=1$, so the equilibrium $\lambda = 0$. As c goes to c^0 from below $\lambda \to 0$, and from (14), (15), and (18) $\lim_{c\uparrow c^0}(1+nm/(1+m))^{-1/2} = e^{-ac^0}$ Hence $\lim_{c\uparrow c^0}(nm/1+m)$ is a finite positive number. Thus from (22) mean trade goes to zero as $c \uparrow c^0$. If the numerator and the denominator of (24) are divided by $(1+m)^2$, then again using the fact that $m/1+m$ has a finite limit gives the result that as $c\uparrow c^0$, $\lambda \to 0$, and variance of trade goes to zero.

(b) By (14), (15), and (18), $nm/(1+m)$ is constant as $n \to \infty$. Further, from remark 2) of Section II, Part I, $\lambda \to 0$ as $n \to \infty$. Hence from (23) and (24), the mean and variance of trade go to zero.

(c) From remark 3) in Section II, Part I, m is constant and λ goes to zero as $\sigma_x^2 \to 0$. Therefore mean trade goes to zero. In (24), note that $(nm + a\sigma_\varepsilon^2/\lambda)^2\sigma_x^2/\sigma_\theta^2 = (nm\sigma_x/\sigma_\theta + (m)^{1/2})^2$ by (16a). Hence the variance of trade goes to zero as $\sigma_x^2 \to 0$.

Note further that $\lambda(X_I - x) + (1-\lambda)(X_U - x) = 0$ implies that no trade will take place as $\lambda \to 1$. Thus, the result that competitive equilibrium is incompatible with informationally efficient markets should be interpreted as meaning that speculative markets where prices reveal a lot of information will be very thin because it will be composed of individuals with very similar beliefs.

IV. On the Possibility of Perfect Markets

In Section II we showed that the price system reveals the signal w_λ^* to traders, where

$$w_\lambda \equiv \theta - \frac{a\sigma_\varepsilon^2}{\lambda}(x - Ex^*)$$

Thus, for given information of informed traders θ, the price system reveals a noisy version of θ. The noise is $(a\sigma_\varepsilon^2/\lambda)(x - Ex^*)$. Uninformed traders learn θ to within a random variable with mean zero and variance $(a\sigma_\varepsilon^2/\lambda)^2\, Var\, x^*$, where σ_ε^2 is the precision of informed traders' information, $Var\, x^*$ is the amount of endowment uncertainty, λ the fraction of informed traders, and a is the degree of absolute risk aversion. Thus, in general the price system does not reveal all

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or $X_I = \dfrac{1+m+nm}{1+m+\lambda nm}$

$$\times\left[x + \frac{(1-\lambda)\left([(m+1)-1](\theta-\bar{\theta}) + \frac{ah}{\lambda}(x-\bar{x})\right)}{ah(1+m+nm)n}\right]$$

$X_I - x =$

$$\frac{(1-\lambda)\left[\left(nm+\frac{ah}{\lambda}\right)(x-\bar{x}) + [(m+1)-1](\theta-\bar{\theta}) + \bar{x}nm\right.}{(1+m+\lambda nm)n}$$

the information about "the true value" of the risky asset. (θ is the true value of the risky asset in that it reflects the best available information about the asset's worth.)

The only way informed traders can earn a return on their activity of information gathering, is if they can use their information to take positions in the market which are "better" than the positions of uninformed traders. "Efficient Markets" theorists have claimed that "at any time prices fully reflect all available information" (see Eugene Fama, p. 383). If this were so then informed traders could not earn a return on their information.

We showed that when the efficient markets hypothesis is true and information is costly, competitive markets break down. This is because when $\sigma_\epsilon^2=0$ or $Var\, x^*=0$, w_λ, and thus price, does reflect all the information. When this happens, each informed trader, because he is in a competitive market, feels that he could stop paying for information and do as well as a trader who pays nothing for information. But all informed traders feel this way. Hence having any positive fraction informed is not an equilibrium. Having no one informed is also not an equilibrium, because then each trader, taking the price as given, feels that there are profits to be made from becoming informed.

Efficient Markets theorists seem to be aware that costless information is a *sufficient* condition for prices to fully reflect all available information (see Fama, p. 387); they are not aware that it is a *necessary* condition. But this is a *reducto ad absurdum*, since price systems and competitive markets are important only when information is costly (see Fredrick Hayek, p. 452).

We are attempting to redefine the Efficient Markets notion, not destroy it. We have shown that when information is very inexpensive, or when informed traders get very precise information, then equilibrium exists and the market price will reveal most of the informed traders' information. However, it was argued in Section III that such markets are likely to be thin because traders have almost homogeneous beliefs.

There is a further conflict. As Grossman (1975, 1977) showed, whenever there are differences in beliefs that are not completely arbitraged, there is an incentive to create a market. (Grossman, 1977, analyzed a model of a storable commodity whose spot price did not reveal all information because of the presence of noise. Thus traders were left with differences in beliefs about the future price of the commodity. This led to the opening of a futures market. But then uninformed traders had two prices revealing information to them, implying the elimination of noise.) But, because differences in beliefs are themselves endogenous, arising out of expenditure on information and the informativeness of the price system, the creation of markets eliminates the differences of beliefs which gave rise to them, and thus causes those markets to disappear. If the creation of markets were costless, as is conventionally assumed in equilibrium analyses, equilibrium would never exist. For instance, in our model, were we to introduce an additional security, say a security which paid

$$z=\begin{cases}1 & \text{if} \quad u>E\theta^*\\ 0 & \text{if} \quad u\leqslant E\theta^*\end{cases}$$

then the demand y for this security by the informed would depend on its price, say q on p and on θ, while the uninformed demand depends only on p and q:

$$\lambda y_I(q,p,\theta)+(1-\lambda)y_u(q,p)=0$$

is the condition that demand equals (supply is zero for a pure security). Under weak assumptions, q and p would convey all the information concerning θ. Thus, the market would be "noiseless" and no equilibrium could exist.

Thus, we could argue as soon as the assumptions of the conventional perfect capital markets model are modified to allow even a slight amount of information imperfection and a slight cost of information, the traditional theory becomes untenable. There *cannot* be as many securities as states of nature. For if there were, competitive equilibrium would not exist.

It is only because of costly transactions and the fact that this leads to there being a limited number of markets, that competitive equilibrium can be established.

We have argued that because information is costly, prices cannot perfectly reflect the information which is available, since if it did, those who spent resources to obtain it would receive no compensation. There is a fundamental conflict between the efficiency with which markets spread information and the incentives to acquire information. However, we have said nothing regarding the social benefits of information, nor whether it is socially optimal to have "informationally efficient markets." We hope to examine the welfare properties of the equilibrium allocations herein in future work.

Appendix A

Here we collect some facts on conditional expectations used in the text. If X^* and Y^* are jointly normally distributed then

$$\text{(A1)}\quad E[X^*|Y^*=Y] = EX^* + \frac{Cov(X^*,Y^*)}{Var(Y^*)}\{Y-EY^*\}$$

$$\text{(A2)}\quad Var[X^*|Y^*=Y] = Var(X^*) - \frac{[Cov(X^*,Y^*)]^2}{Var(Y^*)}$$

(See Paul Hoel, p. 200.) From (A1) note that $E[X^*|Y^*]$ is a function of Y. If the expectation of both sides of (A1) is taken, we see that

$$\text{(A3)}\quad E\{E[X^*|Y^*=Y]\} = EX^*$$

Note that $Var[X^*|Y^*=Y]$ is not a function of Y, as $Var(X^*)$, $Cov(X^*,Y^*)$, and $Var(Y^*)$ are just parameters of the joint distribution of X^* and Y^*.

Two other relevant properties of conditional expectation are

$$\text{(A4)}\quad E\{E[Y^*|F(X^*)]|X^*\} = E[Y^*|F(X^*)]$$

$$\text{(A5)}\quad E\{E[Y^*|X]|F(X^*)\} = E[Y^*|F(X^*)]$$

where $F(\cdot)$ is a given function on the range of X^* (see Robert Ash, p. 260).

Appendix B

PROOF of Theorem 1:

(a) Suppose $\lambda=0$; then (9) becomes

$$\text{(A6)}\quad X_U(P_0(\theta,x),P_0^*) = x$$

Define

$$\text{(A7)}\quad P_0(\theta,x) \equiv \frac{E\theta^* - ax\sigma_u^2}{R}$$

where σ_u^2 is the variance of u. Note that $P_0(\theta^*,x^*)$ is uncorrelated with u^*, as x^* is uncorrelated with u^*. Hence

$$\text{(A8)}\quad E[u^*|P_0^* = P_0(\theta,x)] = Eu^* = E\theta^*$$

$$\text{and}\quad Var[u^*|P_0^* = P_0(\theta,x)] = Var[u^*]$$

Substitution of (A8) in (8) yields

$$\text{(A9)}\quad X_U(P_0^*,P_0(\theta,x)) = \frac{E\theta^* - RP_0(\theta,x)}{a\,Var\,u}$$

Substitution of (A7) in the right-hand side of (A9) yields $X_U(P_0^*(\theta,x),P_0^*) = x$ which was to be shown.

(b) Suppose $0<\lambda\leqslant 1$. Let

$$\text{(A10)}\quad P_\lambda(\theta,x) = \frac{\dfrac{\lambda w_\lambda}{a\sigma_\varepsilon^2} + \dfrac{(1-\lambda)E[u^*|w_\lambda]}{a\,Var[u^*|w_\lambda]} - Ex^*}{R\left[\dfrac{\lambda}{a\sigma_\varepsilon^2} + \dfrac{(1-\lambda)}{a\,Var[u^*|w_\lambda]}\right]}$$

Note that from equations (1), (10), (A1) and (A2):

$$\text{(A11a)}\quad E(u^*|w_\lambda) = E\theta^* + \frac{\sigma_\theta^2}{Var\,w_\lambda}\cdot(w_\lambda - E\theta^*)$$

$$\text{(A11b)}\quad Var(u^*|w_\lambda) = \sigma_\theta^2 + \sigma_\varepsilon^2 - \frac{\sigma_\theta^2}{Var\,w_\lambda}$$

$$\text{(A11c)}\quad Var\,w_\lambda = \sigma_\theta^2 + \left(\frac{a\sigma_\varepsilon^2}{\lambda}\right)^2 Var\,x^*$$

Since $P_\lambda(\theta,x)$ is a linear function of w_λ, it is immediate that $E(u^*|w_\lambda)\equiv E(u^*|P_\lambda)$, $Var(u^*|w_\lambda)=Var(u^*|P_\lambda)$, etc. To see that P_λ^* is an equilibrium, we must show that the following equation holds as an identity in (θ,x), for $P_\lambda(\cdot)$ defined by (A10):

(A12)

$$\lambda\cdot\frac{\theta-RP_\lambda}{a\sigma_\epsilon^2}+(1-\lambda)\frac{E[u^*|w_\lambda]-RP_\lambda}{a\,Var[u^*|w_\lambda]}=x$$

It is immediate from (10) that (A12) holds as an identity in θ and x.

PROOF of Theorem 2:

(a) *Calculation of the expected utility of the informed.* Using the fact that W_{Ii}^λ is normally distributed conditional on $(\bar{X}_i,\theta,x)$

$$\text{(A13)}\quad E\left[V(W_{Ii}^\lambda)|\bar{X}_i,\theta,x\right]=\exp\left[-a\left\{E\left[W_{Ii}^\lambda|\bar{X}_i,\theta,x\right]-\frac{a}{2}Var\left[W_{Ii}^\lambda|\bar{X}_i,\theta,x\right]\right\}\right]$$

Using (8), (12), and the fact that (θ,x) determines a particular P,

$$\text{(A14a)}\quad E\left[W_{Ii}^\lambda|\bar{X}_i,\theta,x\right]=R(W_{0i}-c)+\frac{(E[u^*|\theta]-RP_\lambda)^2}{a\sigma_\epsilon^2}$$

(A14b)

$$Var\left[W_{Ii}^\lambda|\bar{X}_i,\theta,x\right]=\frac{(E[u^*|\theta]-RP_\lambda)^2}{a^2\sigma_\epsilon^2}$$

Substitution of (A14) into (A13) yields

$$\text{(A15)}\quad E\left[V(W_{Ii}^\lambda)|\bar{X}_i,\theta,x\right]=-\exp\left[-aR(W_{0i}-c)-\frac{1}{2\sigma_\epsilon^2}(E[u^*|\theta]-RP_\lambda)^2\right]$$

Note that, as $P_\lambda^*(\cdot)=P_\lambda(\theta,x)$,

$$\text{(A16)}\quad E\left(E\left[V(W_{Ii}^\lambda)|\bar{X}_i,\theta,x\right]|P_\lambda,\bar{X}_i\right)=E\left[V(W_{Ii}^\lambda)|P_\lambda,\bar{X}_i\right]$$

(see (A5)). Note that since W_{0i} is nonstochastic conditional on $(P_\lambda,\bar{X}_i)$, equation (A15) implies

(A17)

$$E\left[V(W_{Ii}^\lambda)|P_\lambda,\bar{X}_i\right]=-\exp\left[-aR(W_{0i}^\lambda-c)\right]\cdot E\left[\left\{\exp\left[-\frac{1}{2\sigma_\epsilon^2}(E[u|\theta]-RP_\lambda)^2\right]\right\}|P_\lambda,\bar{X}_i\right]$$

Note that by Theorem 1, conditioning on w_λ^* is equivalent to conditioning on P_λ^*. Define

$$\text{(A18)}\quad h_\lambda\equiv Var(E[u^*|\theta]|w_\lambda)=Var(\theta|w_\lambda),h_0\equiv\sigma_\epsilon^2\equiv h$$

$$\text{(A19)}\quad Z\equiv\frac{E[u^*|\theta]-RP_\lambda}{\sqrt{h_\lambda}}$$

Using (3) and (A18), equation (A17) can be written as

$$\text{(A20)}\quad E\left[V(W_{Ii}^\lambda)|P_\lambda,\bar{X}_i\right]=e^{ac}V(RW_{0i})E\left[\exp\left[-\frac{h_\lambda}{2\sigma_\epsilon^2}Z^2\right]|w_\lambda\right]$$

since $\bar{X}_i$ and w_λ are independent. Conditional on w_λ, P_λ is nonstochastic and $E[u^*|\theta]$ is normal. Hence conditional on $w_\lambda,(Z^*)^2$ has a noncentral *chi*-square distribution (see C. Rao, p. 181). Then for $t>0$ the moment generating function for $(Z^*)^2$ can be written

$$\text{(A21)}\quad E\left[e^{-tZ^2}|w_\lambda\right]=\frac{1}{\sqrt{1+2t}}\exp\left[\frac{-(E[Z|w_\lambda])^2t}{1+2t}\right]$$

Note that $E[u^*|\theta]= E[u^*|\theta,x]$. Hence

$$\text{(A22)}\quad E\big[E[u^*|\theta]|w_\lambda\big]=E[u^*|w_\lambda] = E\theta^* + \frac{\sigma_\theta^2}{Var\,w_\lambda}(w_\lambda - E\theta^*)$$

since w_λ is just a function of (θ,x). Therefore

$$\text{(A23)}\quad E[Z^*|w_\lambda]=\frac{E[u^*|w_\lambda]-RP_\lambda}{\sqrt{h_\lambda}}$$

Since $u=\theta+\varepsilon$

$$\text{(A24)}\quad Var(u^*|w_\lambda)=\sigma_\varepsilon^2+Var(\theta^*|w_\lambda)=\sigma_\varepsilon^2+h_\lambda$$

The nondegeneracy assumptions on (x^*, ε^*, u^*) imply $h_\lambda>0$. Set $t=(h_\lambda/2\sigma_\varepsilon^2)$; and evaluate (A21) using (A23) and (A24):

$$\text{(A25)}\quad E\left[\exp\left[-\frac{h_\lambda}{2\sigma_\varepsilon^2}Z^2\right]|w_\lambda\right]=\sqrt{\frac{Var(u^*|\theta)}{Var(u^*|w_\lambda)}}\cdot\exp\left(\frac{-(E(u^*|w_\lambda)-RP_\lambda)^2}{2\,Var(u^*|w_\lambda)}\right)$$

This permits the evaluation of (A20).

(b) *Calculation of expected utility of the uninformed.* Equations (8), (5), and the normality of W_{Ui}^λ conditional on w_λ can be used to show, by calculations parallel to (A13)–(A25), that

$$\text{(A26)}\quad E\left[V(W_{Ui}^\lambda)|w_\lambda,\bar{X}_i\right] = V(RW_{0i})\exp\left(\frac{-(E(u^*|w_\lambda)-RP_\lambda)^2}{2\,Var(u^*|w_\lambda)}\right)$$

Hence

$$\text{(A27)}\quad E\left[V(W_{Ii}^\lambda)|w_\lambda,\bar{X}_i\right]-E\left[V(W_{Ui}^\lambda)|w_\lambda,\bar{X}_i\right] = \left[e^{ac}\sqrt{\frac{Var(u^*|\theta)}{Var(u^*|w_\lambda)}}-1\right]\times E\left[V(W_{Ui}^\lambda)|w_\lambda,\bar{X}_i\right]$$

Taking expectations of both sides of (A27) yields:

$$\text{(A28)}\quad E\left[V(W_{Ii}^\lambda)\right]-E\left[V(W_{Ui}^\lambda)\right] = \left[e^{ac}\sqrt{\frac{Var(u^*|\theta)}{Var(u^*|w_\lambda)}}-1\right]EV(W_{Ui}^\lambda)$$

Equation (13) follows immediately from (A28).

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